Please check the examination details below before entering your candidate information

| Candidate surname | Other names |
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Centre Number

Candidate Number

# Pearson Edexcel Level 3 GCE

## Tuesday 13 June 2023

Afternoon (Time: 2 hours)

**Paper reference** **9MA0/02**

# Mathematics

## Advanced

## PAPER 2: Pure Mathematics 2

**You must have:**
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

**Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question*.

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over*

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Pearson

**1.**

$$f(x) = x^3 + 2x^2 - 8x + 5$$

(a) Find $f''(x)$

**(2)**

(b) (i) Solve $f''(x) = 0$

(ii) Hence find the range of values of $x$ for which $f(x)$ is concave.

**(2)**

(a) We need to differentiate twice ★ simple differentiation: $\frac{dy}{dx} = nx^{n-1}$

$f'(x) = 3x^2 + 4x - 8$ M1

$f''(x) = 6x + 4$ A1

(b) i. equate to 0:

$6x + 4 = 0$ $x = -\frac{2}{3}$ B1

ii. For $f(x)$ to be concave it must be before the point of inflection

$\therefore x < -\frac{2}{3}$ B1

**Question 1 continued**

**(Total for Question 1 is 4 marks)**

**2.** A sequence $u_1, u_2, u_3 \dots$ is defined by

$$u_1 = 35$$

$$u_{n+1} = u_n + 7\cos\left(\frac{n\pi}{2}\right) - 5(-1)^n$$

(a) (i) Show that $u_2 = 40$

(ii) Find the value of $u_3$ and the value of $u_4$

**(3)**

Given that the sequence is periodic with order 4

(b) (i) write down the value of $u_5$

(ii) find the value of $\sum_{r=1}^{25} u_r$

**(3)**

(a) i. $u_2 = u_1 + 7\cos\left(\frac{1 \times \pi}{2}\right) - 5(-1)^1$ substitute n or $u_n$

$= 35 + 7\cos\left(\frac{\pi}{2}\right) + 5$

$= 35 + 7(0) + 5$

$= 40$ hence shown B1

ii. $u_3 = 40 + 7\cos\left(\frac{2 \times \pi}{2}\right) - 5(-1)^2$ M1

$= 40 + 7\cos(\pi) - 5$

$= 28 = u_3$

$u_4 = 28 + 7\cos\left(\frac{3\pi}{2}\right) - 5(-1)^3$

$= 28 + 7(0) + 5$

$= 33 = u_4$ A1

(b) i. $u_5 = 35$ B1

this will repeat 6 times up to term 24

ii. $\frac{25}{4} = 6\frac{1}{4}$.

$\sum_{r=1}^{25} u_r = 6 \times (35 + 40 + 28 + 33) + 35$ M1

$= 851$ A1

since it's periodic we know what term 25 will be

**Question 2 continued**

**(Total for Question 2 is 6 marks)**

3. Given that

$$\log_2(x + 3) + \log_2(x + 10) = 2 + 2\log_2 x$$

(a) show that

$$3x^2 - 13x - 30 = 0$$

**(3)**

(b) (i) Write down the roots of the equation

$$3x^2 - 13x - 30 = 0$$

(ii) Hence state which of the roots in part (b)(i) is not a solution of

$$\log_2(x + 3) + \log_2(x + 10) = 2 + 2\log_2 x$$

giving a reason for your answer.

**(2)**

(a) We need to use log laws

Log Law used

$\log_2(x+3) + \log_2(x+10) = 2 + 2\log_2 x$ M1 $\log a + \log b = \log ab$

$\log_2(x+3)(x+10) = 2 + 2\log_2 x$ $a\log b = \log b^a$

$\log_2(x^2+13x+30) = 2 + \log_2 x^2$

$\log_2(x^2+13x+30) - \log_2 x^2 = 2$ $\log a - \log b = \log \frac{a}{b}$

$\log_2\left(\frac{x^2+13x+30}{x^2}\right) = 2$

$\frac{x^2+13x+30}{x^2} = 4$

$x^2+13x+30 = 4x^2$ dM1

$0 = 3x^2 - 13x - 30$ hence shown A1

(b) i. $0 = (3x+5)(x-6)$

$x = -\frac{5}{3}$, $x = 6$ B1

ii. $x \neq -\frac{5}{3}$ since logs can't take negative values → $\log_2(-\frac{5}{3})$ isn't real. B1

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**Question 3 continued**

**(Total for Question 3 is 5 marks)**





Turn over

4. Coffee is poured into a cup.

The temperature of the coffee, $H$ °C, $t$ minutes after being poured into the cup is modelled by the equation

$$H = Ae^{-Bt} + 30$$

where $A$ and $B$ are constants.

$t=0$

Initially, the temperature of the coffee was 85 °C.

(a) State the value of $A$.

**(1)**

$t=0$ $\frac{dH}{dt}$

Initially, the coffee was cooling at a rate of 7.5 °C per minute.

(b) Find a complete equation linking $H$ and $t$, giving the value of $B$ to 3 decimal places.

**(3)**

(a) at $t=0$, $H=85$ $\therefore$ $85 = Ae^{0} + 30$

$85 = A + 30 \longrightarrow A = 55$ B1

(b) Differentiate:

$\frac{dH}{dt} = -55Be^{-Bt}$ M1

at $t=0$, $\frac{dH}{dt} = -7.5$:

$-7.5 = -55Be^{0}$

$B = \frac{-7.5}{-55} \longrightarrow B = \frac{3}{22} = 0.136$ M1

$\therefore$ $H = 55e^{-0.136t} + 30$ A1

**Question 4 continued**

**(Total for Question 4 is 4 marks)**

**5.** The curve $C$ has equation $y = \mathrm{f}(x)$

The curve

- passes through the point $P(3, -10)$
- has a turning point at $P$

Given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^3 - 9x^2 + 5x + k$$

where $k$ is a constant,

(a) show that $k = 12$

**(2)**

(b) Hence find the coordinates of the point where $C$ crosses the $y$-axis. Remember to give in the form (a, b)

**(3)**

(a) "turning point at P" $\therefore \frac{dy}{dx} = 0$ at $x = 3$

$\frac{dy}{dx} = 0 = 2(3)^3 - 9(3)^2 + 5(3) + k$ M1

$0 = 54 - 81 + 15 + k$

$k = 12$ hence shown A1

(b) To get the equation for C we need to integrate

$\int 2x^3 - 9x^2 + 5x + 12\,dx$

$= \frac{2}{4}x^4 - \frac{9}{3}x^3 + \frac{5}{2}x^2 + 12x$ M1

★ Simple Integration: $\int x^n\,dx = \frac{1}{n}x^{n+1} + c$

$= \frac{1}{2}x^4 - 3x^3 + \frac{5}{2}x^2 + 12x + c$

to get c use P(3, −10):

$-10 = \frac{1}{2}(3)^4 - 3(3)^3 + \frac{5}{2}(3)^2 + 12(3) + c$

$-10 = \frac{81}{2} - 81 + \frac{45}{2} + 36 + c$ dM1

$c = -28$ — this is the y-intercept

$\therefore (0, -28)$ A1

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**Question 5 continued**

**(Total for Question 5 is 5 marks)**

**6.** Relative to a fixed origin $O$,

- $A$ is the point with position vector $12\mathbf{i}$ $\begin{pmatrix}12\\0\end{pmatrix}$
- $B$ is the point with position vector $16\mathbf{j}$ $\begin{pmatrix}0\\16\end{pmatrix}$
- $C$ is the point with position vector $(50\mathbf{i} + 136\mathbf{j})$ $\begin{pmatrix}50\\136\end{pmatrix}$
- $D$ is the point with position vector $(22\mathbf{i} + 24\mathbf{j})$ $\begin{pmatrix}22\\24\end{pmatrix}$

(a) Show that $AD$ is parallel to $BC$.

**(2)**

Points $A$, $B$, $C$ and $D$ are used to model the vertices of a running track in the shape of a quadrilateral.

Runners complete one lap by running along all four sides of the track.

The lengths of the sides are measured in metres.

Given that a particular runner takes exactly 5 minutes to complete 2 laps,

(b) calculate the average speed of this runner, giving the answer in kilometres per hour.

**(4)**

(a) Calculate $\vec{AD}$ and $\vec{BC}$:

$$\vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix}22\\24\end{pmatrix} - \begin{pmatrix}12\\0\end{pmatrix} = \begin{pmatrix}10\\24\end{pmatrix}$$

$$\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix}50\\136\end{pmatrix} - \begin{pmatrix}0\\16\end{pmatrix} = \begin{pmatrix}50\\120\end{pmatrix}$$ M1

Now we need to show that $\vec{AD}$ and $\vec{BC}$ are multiples of eachother

$$\begin{pmatrix}10\\24\end{pmatrix} \times 5 = \begin{pmatrix}50\\120\end{pmatrix}$$

$$\therefore \ 5\vec{AD} = \vec{BC}$$ A1

Since $\vec{BC}$ is a multiple of $\vec{AD}$, the two are parallel

Question 6 continued

(b) We need the magnitudes of the vectors $\vec{AB}, \vec{BC}, \vec{CD}$ and $\vec{AD}$.

We found vectors $\vec{BC}$ and $\vec{AD}$ in (a)

Let's find vectors $\vec{AB}$ and $\vec{CD}$:

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 0 \\ 16 \end{pmatrix} - \begin{pmatrix} 12 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 16 \end{pmatrix}$$

$$\vec{CD} = \vec{OD} - \vec{OC} = \begin{pmatrix} 22 \\ 24 \end{pmatrix} - \begin{pmatrix} 50 \\ 136 \end{pmatrix} = \begin{pmatrix} -28 \\ -112 \end{pmatrix}$$

Now find the magnitudes $|\vec{AB}|, |\vec{BC}|, |\vec{CD}|, |\vec{AD}|$ using Pythagoras' Theorem

$$|\vec{AB}| = \sqrt{12^2 + 16^2} = 20$$

$$|\vec{BC}| = \sqrt{50^2 + 120^2} = 130$$

$$|\vec{CD}| = \sqrt{28^2 + 112^2} = 28\sqrt{17}$$

$$|\vec{AD}| = \sqrt{10^2 + 24^2} = 26$$

M1A1

Now we can get the length of 1 lap:

$$20 + 130 + 26 + 28\sqrt{17} = 176 + 28\sqrt{17}$$

∴ 2 laps: $2(176 + 28\sqrt{17})$ m

To get average speed use the formula:

$$\text{av. speed} = \frac{\text{distance}}{\text{time}}$$

convert $2(176+28\sqrt{17})$m to km by dividing by 1000 and the 5minutes to hours by dividing by 60:

$$\frac{2(176+28\sqrt{17})}{1000} = \frac{176+28\sqrt{17}}{500}\text{ km} \qquad \frac{5}{60} = \frac{1}{12}\text{ hours}$$

Substitute into formula:

$$\text{av. speed} = \frac{\frac{176+28\sqrt{17}}{500}\text{ km}}{\frac{1}{12}\text{ h}}$$

dM1

$$= 6.99\text{ km/h to 3sf.}$$

A1

**Question 6 continued**

**Question 6 continued**

**(Total for Question 6 is 6 marks)**



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**7.** **In this question you must show all stages of your working.**

**Solutions relying on calculator technology are not acceptable.**

A curve has equation

$$x^3 + 2xy + 3y^2 = 47$$

(a) Find $\frac{dy}{dx}$ in terms of $x$ and $y$

**(4)**

The point $P(-2, 5)$ lies on the curve.

(b) Find the equation of the normal to the curve at $P$, giving your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers to be found.

**(3)**

(a) We need to use implicit differentiation

★Implicit: when differentiating y, multiply by $\frac{dy}{dx}$

★Product Rule:

$\frac{d}{dx}(uv) = uv' + u'v$

$u = 2x \quad u' = 2$

$v = y \quad v' = \frac{dy}{dx}$

$2y + 2x\frac{dy}{dx}$ B1

$x^3 + 2xy + 3y^2 = 47$

$3x^2 + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ M1

$2x\frac{dy}{dx} + 6y\frac{dy}{dx} = -3x^2 - 2y$

$(2x+6y)\frac{dy}{dx} = -3x^2 - 2y$ M1

$\frac{dy}{dx} = \frac{-3x^2 - 2y}{2x+6y}$ A1

(b) To get the gradient at a specific point, substitute the point's coordinates

$\frac{dy}{dx} = -\frac{3(-2)^2+2(5)}{2(-2)+6(5)} = -\frac{12+10}{30-4}$ M1

$= -\frac{22}{26} = -\frac{11}{13}$ ← gradient of the tangent

To get the gradient of the normal use $m_1 \times m_2 = -1$ to get the perpendicular gradient (★Remember: the normal is perpendicular to the curve)

$-\frac{11}{13} \times m_2 = -1 \quad m_2 = \frac{13}{11}$

Use $y - y_1 = m(x - x_1)$ to get the equation of the line

$y - 5 = \frac{13}{11}(x - -2)$ dM1

$11y - 55 = 13x + 26$

$0 = 13x - 11y + 81$ A1

**Question 7 continued**

**(Total for Question 7 is 7 marks)**

**8.** (a) Express $2\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R$ and $\alpha$ are constants,

$R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of $R$ and give the value of $\alpha$ in radians to 3 decimal places.

**(3)**

∴ common difference d

The first three terms of an arithmetic sequence are

1 $\quad$ 2 $\quad$ 3

$$\cos x \qquad \cos x + \sin x \qquad \cos x + 2\sin x \qquad x \neq n\pi$$

Given that $S_9$ represents the sum of the first 9 terms of this sequence as $x$ varies,

(b) (i) find the exact maximum value of $S_9$

(ii) deduce the smallest positive value of $x$ at which this maximum value of $S_9$ occurs.

**(3)**

(a) Method 1: LONGER

rewrite $R\cos(\theta - a)$ using addition formulae and compare coefficients

$\cos(\theta - a) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$

$R\cos(\theta - a) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

compare coefficients:

$2\cos\theta \Rightarrow R\cos\alpha = 2$

$8\sin\theta \Rightarrow R\sin\alpha = 8 \quad \therefore R\sin\alpha = 8$

Now divide the two to get the identity $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

R's cancel out $\quad \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{8}{2} \rightarrow \tan\alpha = 4 \quad \alpha = 1.326$ rad to 3 dp A1

$\tan^{-1}(4) = \alpha$ M1

Substitute into $R\cos\alpha = 2$

$R = \dfrac{2}{\cos(1.326\ldots)} = 2\sqrt{17}$ A1 $\rightarrow 2\sqrt{17}\cos(\theta - 1.326)$

Method 2: SHORTER

Pythagoras' Theorem to get :

$\sqrt{2^2 + 8^2} = R \rightarrow = 2\sqrt{17}$ A1

Now use addition formula and compare : $\rightarrow 2\sqrt{17}\cos(\theta - 1.326)$

$\cos(\theta - \alpha) = 2\cos\theta\cos\alpha + 8\sin\theta\sin\alpha$

$2\sqrt{17}\cos\alpha = 2$

M1 $\cos\alpha = \dfrac{2}{2\sqrt{17}} \quad \cos^{-1}\left(\dfrac{2}{2\sqrt{17}}\right) = \alpha \rightarrow \alpha = 1.326$ rad to 3dp A1

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**Question 8 continued**

(b)i. Formula for sum of arithmetic sequence:

$S_n = \frac{n}{2}[2a + d(n-1)]$

$a = \cos x$

$d = \sin x$

$n = 9$

$S_9 = \frac{9}{2}[2\cos x + \sin x(9-1)]$

$= 4.5[2\cos x + 8\sin x]$

$= 4.5[2\sqrt{17}\cos(x - 1.326)]$ from (a)

For $S_9$ to be maximum, $\cos(x-1.326)$ must be maximum $\therefore = 1$

$\therefore S_9 = 4.5 \times 2\sqrt{17}(1)$ M1

$= 9\sqrt{17}$ A1

ii. Equate $\cos(x-1.326) = 1$ to get the value of $x$ for which cos was maximum

$x - 1.326 = \cos^{-1}(1)$ 0

$x = 1.33$ to 3sf. B1



Turn over

P 7 2 8 0 5 A 0 1 9 4 8

**Question 8 continued**

**Question 8 continued**

**(Total for Question 8 is 6 marks)**

**9.** The curve $C$ has parametric equations

$$x = t^2 + 6t - 16 \qquad y = 6\ln(t+3) \qquad t > -3$$

(a) Show that a Cartesian equation for $C$ is

$$y = A\ln(x + B) \qquad x > -B$$

where $A$ and $B$ are integers to be found.

**(3)**

The curve $C$ cuts the $y$-axis at the point $P$

(b) Show that the equation of the tangent to $C$ at $P$ can be written in the form

$$ax + by = c\ln 5$$

where $a$, $b$ and $c$ are integers to be found.

**(4)**

(a) Method 1 Complete the square for x:

$x = (t+3)^2 - 9 - 16$

$x = (t+3)^2 - 25$ M1

$x + 25 = (t+3)^2$

M1 $(x+25)^{\frac{1}{2}} = (t+3)$ See that this shows up in the y=... equation

Substitute:

$y = 6\ln(x+25)^{\frac{1}{2}}$ use log law: $\ln a^b = b\ln a$

$y = 3\ln(x+25)$ A1

Method 2 manipulate $y = 6\ln(t+3)$ first

$y = 6\ln(t+3)$ use log law: $\ln a^b = b\ln a$

M1 $y = 3\ln(t+3)^2$ ← expand this

$= 3\ln(t^2+6t+9)$ → this looks similar to $x = t^2+6t-16$

M1 $= 3\ln(x+16+9)$ ← substitute $t^2+6t = x+16$

$y = 3\ln(x+25)$ A1

Method 3 make t the subject of y=...

$y = 6\ln(t+3)$

$\frac{y}{6} = \ln(t+3)$

$e^{\frac{y}{6}} = e^{\ln(t+3)}$

$e^{\frac{y}{6}} = t+3$

$t = e^{\frac{y}{6}} - 3$ M1

Substitute into x = ... :

$x = (e^{\frac{y}{6}} - 3)^2 + 6(e^{\frac{y}{6}} - 3) - 16$ M1

**Question 9 continued**

Now we need to make y the subject:

$x = e^{2(\frac{y}{6})} - 6e^{\frac{y}{6}} + 9 + 6e^{\frac{y}{6}} - 18 - 16$

$x = e^{\frac{y}{3}} - 25$

$x + 25 = e^{\frac{y}{3}}$

apply ln to both sides

$\ln e^{x} = x$

$\ln(x+25) = \frac{y}{3}$

$y = 3\ln(x+25)$ A1

(b) At point P: $x = 0$: $x = t^2 + 6x - 16$

$x = (t+8)(t-2)$

$t = 2$ ~~$t = -8$~~ $t > -3$ given domain

$y = 6\ln(2+3)$

$y = 6\ln 5$ B1

∴ Point $P(0, 6\ln 5)$

$y = 3\ln(x+25)$

$\frac{dy}{dx} = \frac{3}{x+25}$

★ $\frac{d}{dx}(\ln x) = \frac{1}{x}$

★chain rule: multiply by the derivative of the bracket!

Substitute $x=0$ to get m:

$\frac{dy}{dx} = \frac{3}{25} = m$ M1

Use $y - y_1 = m(x - x_1)$ to get the equation of the line

$y - 6\ln 5 = \frac{3}{25}x$ dM1

$25y - 150\ln 5 = 3x$

$25y - 3x = 150\ln 5$ A1

$a = 25$ $b = -3$ $c = 150$

Question 9 continued

**Question 9 continued**

**(Total for Question 9 is 7 marks)**

**10.** $$f(x) = \frac{3kx - 18}{(x+4)(x-2)} \quad \text{where } k \text{ is a positive constant}$$

(a) Express f(x) in partial fractions in terms of $k$.

**(3)**

(b) Hence find the exact value of $k$ for which

$$\int_{-3}^{1} f(x)\, dx = 21$$

**(4)**

(a) $\frac{3kx-18}{(x+4)(x-2)} \equiv \frac{A}{(x+4)} + \frac{B}{(x-2)}$ → For partial fractions we want to separate the denominator and find the 2 new numerators

$3kx - 18 = A(x-2) + B(x+4)$ B1

Substitute values:

$x = 2$ $6k - 18 = A(2-2) + 6B$ (→ 0) ⟶ $B = k-3$

$x = -4$ $-12k - 18 = A(-4-2) + B(-4+4)$ (→ 0) ⟶ $-12k - 18 = -6A$ ⟶ $A = 2k+3$ M1

Substitute A and B into $\frac{A}{(x+4)} + \frac{B}{(x-2)}$:

$$\frac{2k+3}{(x+4)} + \frac{k-3}{(x-2)} \quad \text{A1}$$

(b) $$\int_{-3}^{1} f(x)\, dx = 21$$

Use what we found in (a)

$$\int_{-3}^{1} \frac{2k+3}{(x+4)} + \frac{k-3}{(x-2)}\, dx = 21$$

↓ Integrate

★ $\int \frac{1}{x}\, dx = \ln x$ and use Reverse chain rule: divide by the derivative of the bracket

$$\Big[(2k+3)\ln|x+4| + (k-3)\ln|x-2|\Big]_{-3}^{1} = 21 \quad \text{M1A1}$$

absolute value as ln can't take negatives

Substitute limits: $\Big((2k+3)\ln(1+4) + (k-3)\ln|1-2|\Big) - \Big((2k+3)\ln|-3+4| + (k-3)\ln|-3-2|\Big) = 21$

Solve for k: $((2k+3)\ln(5) + (k-3)\ln(1)) - ((2k+3)\ln(1) + (k-3)\ln(5)) = 21$ ($\ln(1) = 0$)

$(2k+3)\ln 5 - (k-3)\ln 5 = 21$

$\ln 5\,[2k+3-k+3] = 21$

$\ln 5\,(k+6) = 21$

$k + 6 = \frac{21}{\ln 5}$ dM1

$$k = \frac{21}{\ln 5} - 6 \quad \text{A1}$$

**Question 10 continued**

**Question 10 continued**

**Question 10 continued**

**(Total for Question 10 is 7 marks)**

**11.**

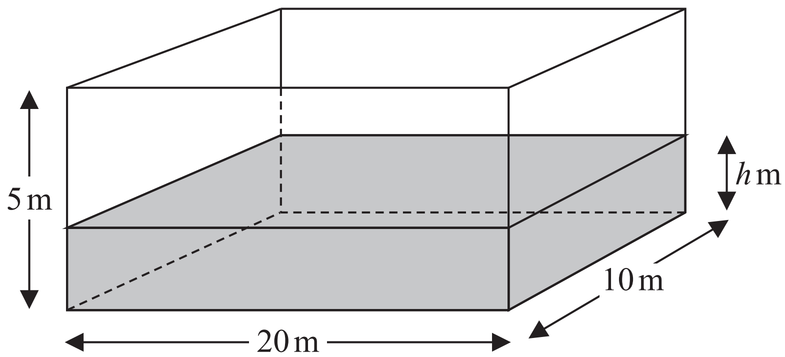


**Figure 1**

A tank in the shape of a cuboid is being filled with water.

The base of the tank measures 20 m by 10 m and the height of the tank is 5 m, as shown in Figure 1.

At time $t$ minutes after water started flowing into the tank the height of the water was $h$ m and the volume of water in the tank was $V$ m$^3$

In a model of this situation

- the sides of the tank have negligible thickness
- the rate of change of $V$ is inversely proportional to the square root of $h$

(a) Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\lambda}{\sqrt{h}}$$

where $\lambda$ is a constant.

**(3)**

Given that

- initially the height of the water in the tank was 1.44 m
- exactly 8 minutes after water started flowing into the tank the height of the water was 3.24 m

(b) use the model to find an equation linking $h$ with $t$, giving your answer in the form

$$h^{\frac{3}{2}} = At + B$$

where $A$ and $B$ are constants to be found.

**(5)**

(c) Hence find the time taken, from when water started flowing into the tank, for the tank to be completely full.

**(2)**





P 7 2 8 0 5 A 0 3 0 4 8

**Question 11 continued**

(a) "The rate of change of V is inversely proportional to the square root of h"

$\therefore \frac{dV}{dt} \propto \frac{1}{\sqrt{h}} \rightarrow \frac{dV}{dt} = \frac{k}{\sqrt{h}}$ — k: constant; $\propto$: proportionality sign

V at t = 20×10×h = 200h

$\frac{dV}{dh} = 200$ B1

★ simple differentiation: $\frac{dy}{dx} = nx^{n-1}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ (cancel out)

$= \frac{k}{\sqrt{h}} \times \frac{1}{200} = \frac{\frac{1}{200}k}{\sqrt{h}}$ M1

$\therefore \frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ $\lambda = \frac{1}{200}k$ A1

(b) We need to separate variables

$\frac{dh}{dt} = \frac{\lambda}{\sqrt{h}}$ (treat as a fraction)

$\int \sqrt{h}\, dh = \int \lambda\, dt$ M1

↓ integrate

★ Simple Integration: $\int x^n\, dx = \frac{1}{n}x^{n+1} + c$

$\frac{2}{3}h^{\frac{3}{2}} = \lambda t + c$

$\frac{2}{3}h^{\frac{3}{2}} = \lambda t + c$ A1

Substitute t=0, h=1.44m to get +c:

$\frac{2}{3}(1.44)^{\frac{3}{2}} = \lambda(0) + c \longrightarrow c = \frac{144}{125}$ dM1

Substitute t=8, h=3.24m to get $\lambda$:

$\frac{2}{3}(3.24)^{\frac{3}{2}} = 8\lambda + \frac{144}{125}$

$\frac{1}{8}\left(\frac{2}{3}(3.24)^{\frac{3}{2}} - \frac{144}{125}\right) = \lambda \longrightarrow \lambda = \frac{171}{500}$ ddM1

$\therefore h^{\frac{3}{2}} = \left(\frac{3}{2} \times \frac{171}{500}\right)t + \frac{3}{2}\left(\frac{144}{125}\right)$

$h^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$ A1

(c) Substitute h=5:

$5^{\frac{3}{2}} = \frac{513}{1000}t + \frac{216}{125}$ M1

$t = \frac{1000}{513}\left(5^{\frac{3}{2}} - \frac{216}{125}\right) \longrightarrow t = 18.4$ minutes A1

DO NOT WRITE IN THIS AREA



P 7 2 8 0 5 A 0 3 1 4 8

**Question 11 continued**





P 7 2 8 0 5 A 0 3 2 4 8

**Question 11 continued**

**(Total for Question 11 is 10 marks)**

**12.**

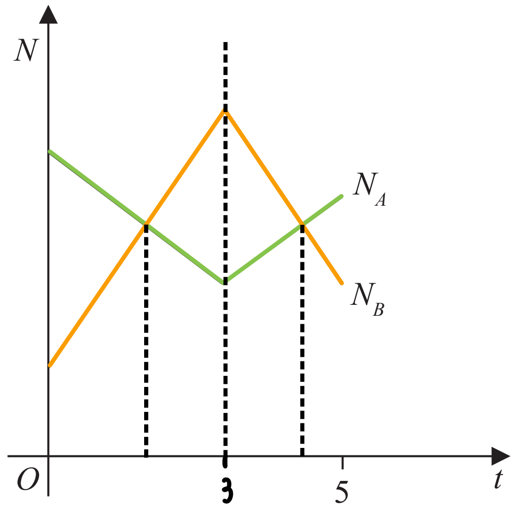


**Figure 2**

The number of subscribers to two different music streaming companies is being monitored.

The number of subscribers, $N_A$, in thousands, to **company A** is modelled by the equation

$$N_A = |t - 3| + 4 \qquad t \geqslant 0$$

where $t$ is the time in years since monitoring began.

The number of subscribers, $N_B$, in thousands, to **company B** is modelled by the equation

$$N_B = 8 - |2t - 6| \qquad t \geqslant 0$$

where $t$ is the time in years since monitoring began.

Figure 2 shows a sketch of the graph of $N_A$ and the graph of $N_B$ over a 5-year period.

**Use the equations of the models to answer parts (a), (b), (c) and (d).**

(a) Find the initial difference between the number of subscribers to **company A** and the number of subscribers to **company B**.

**(2)**

When $t = T$ **company A** reduced its subscription prices and the number of subscribers increased.

(b) Suggest a value for $T$, giving a reason for your answer.

**(2)**

(c) Find the range of values of $t$ for which $N_A > N_B$ giving your answer in set notation.

**(5)**

(d) State a limitation of the model used for **company B**.

**(1)**





P 7 2 8 0 5 A 0 3 4 4 8

Question 12 continued

(a) We need to find $N_A$ and $N_B$ at $t=0$:

$N_A = |0-3| + 4$
$= 3 + 4$
$= 7$ thousand — remember $N_A$ and $N_B$ are given in thousands!

$N_B = 8 - |2(0) - 6|$
$= 8 - 6$
$= 2$ thousand M1

the difference is ∴ $7 - 2 =$ 5000 subscribers A1

(b) We need the point when the subscribers of A started increasing again, which is the point when A had the minimum number of subscribers. B1

This occurs at $T = 3$ B1

(c) We need to find the intersection points of $N_A$ and $N_B$.

Equate $N_A$ and $N_B$:

☆ While $N_A$ is decreasing, see that $N_B$ is increasing and vice versa. Hence we need to always "flip" the brackets depending on what sign the t's have (+ or −)

Case I ($t < 3$)

$|t-3| + 4 = 8 - |2t+6|$

$N_A$ is decreasing ∴ $t_A$ must have negative coefficient.

$N_B$ is increasing ∴ $t_A$ must have positive coefficient.

$(3-t) + 4 = 8 - (6-2t)$ — flip so −−2t and +t is positive!

flip so t is negative! $3 - t + 4 = 8 + 2t - 6$

$7 - t = 2 + 2t$

$5 = 3t$

$t = \frac{5}{3}$ M1A1

Case II ($t > 3$)

$|t-3| + 4 = 8 - |2t-6|$

$N_B$ is decreasing ∴ $t_B$ must have negative coefficient.

$N_A$ is increasing ∴ $t_A$ must have positive coefficient.

$(t-3) + 4 = 8 - (2t-6)$ ☆ don't flip either bracket

$t + 1 = 14 - 2t$ B1

$3t = 13$

$t = \frac{13}{3}$

So for $N_A > N_B$ to be true, $t < \frac{5}{3}$ or $t > \frac{13}{3}$ A1

We have to give this in set notation

$\{t \in \mathbb{R} \mid t < \frac{5}{3}\} \cup \{t \in \mathbb{R} \mid t > \frac{13}{3}\}$ A1

(d) When $t > 7$, the number of subscribers would become negative, which is unrealistic B1

Question 12 continued

**Question 12 continued**

**(Total for Question 12 is 10 marks)**

**13.** **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(a) Find the first three terms, in ascending powers of $x$, of the binomial expansion of

up to $x^2$

$$(3 + x)^{-2}$$

writing each term in simplest form.

**(4)**

(b) Using the answer to part (a) and using algebraic integration, estimate the value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,dx$$

It's an estimate as we'll use the first 3 terms of an expansion

giving your answer to 4 significant figures.

**(4)**

(c) Find, using algebraic integration, the exact value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,dx$$

giving your answer in the form $a \ln b + c$, where $a$, $b$ and $c$ are constants to be found.

**(5)**

(a) Get this formula from the formula booklet:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

to get this form factor out 3:

$$(3+x)^{-2} = (3)^{-2}\left(1+\frac{x}{3}\right)^{-2} \quad \text{M1}$$

the required form to apply the formula.

$$(3)^{-2}\left(1+\frac{x}{3}\right)^{-2} = \frac{1}{9}\left[1 + (-2)\left(\frac{x}{3}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{x}{3}\right)^2 + \ldots\right] \quad \text{M1A1}$$

$$= \frac{1}{9}\left[1 - \frac{2}{3}x + \frac{(-2)(-3)}{2}\left(\frac{x^2}{9}\right)\right]$$

$$= \frac{1}{9}\left(1 - \frac{2}{3}x + \frac{6}{2}\times\frac{x^2}{9}\right)$$

$$= \frac{1}{9}\left(1 - \frac{2}{3}x + \frac{x^2}{3}\right)$$

$$= \frac{1}{9} - \frac{2}{27}x + \frac{x^2}{27} \quad \text{A1}$$

DO NOT WRITE IN THIS AREA

Question 13 continued

(b) From Part (a):

$$(3+x)^{-2} = \frac{1}{(3+x)^2}$$

Index Law: $x^{-a} = \frac{1}{x^a}$

Hence we can rewrite the integral:

what we got in (a)

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,dx = \int_{0.2}^{0.4} 6x \cdot \left(\frac{1}{9} - \frac{2}{27}x + \frac{x^2}{27}\right) dx$$

$$= \int_{0.2}^{0.4} \frac{6x}{9} - \frac{12}{27}x^2 + \frac{6x^3}{27}\,dx$$

$$= \int_{0.2}^{0.4} \frac{2x}{3} - \frac{4}{9}x^2 + \frac{2}{9}x^3\,dx \quad \text{M1}$$

★ Simple Integration: $\int x^n\,dx = \frac{1}{n}x^{n+1} + c$

remember no $+c$ when you have limits!

↓ Integrate

$$= \left[\frac{\frac{2}{3}}{2}x^2 - \frac{\frac{4}{9}}{3}x^3 + \frac{\frac{2}{9}}{4}x^4\right]_{0.2}^{0.4}$$

$$= \left[\frac{1}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{18}x^4\right]_{0.2}^{0.4}$$

$$= \left[\frac{x^2}{3} - \frac{4x^3}{27} + \frac{x^4}{18}\right]_{0.2}^{0.4} \quad \text{A1}$$

substitute limits

$$= \left(\frac{0.4^2}{3} - \frac{4(0.4)^3}{27} + \frac{0.4^4}{18}\right) - \left(\frac{0.2^2}{3} - \frac{4(0.2)^3}{27} + \frac{(0.2)^4}{18}\right) \quad \text{dM1}$$

$$= \frac{223}{6750} = 0.03304 \quad \text{A1}$$

this is an estimate as we only used the first 3 terms of the expansion

(c) $\int_{0.2}^{0.4} 6x(3+x)^{-2}\,dx$

we see this is a multiplication ∴ we have to use Integration by Parts.

Method 1 Formula

★ Integration by Parts: $\int vu'\,dx = vu - \int uv'\,dx$

OR

SDI table method

$u = -(3+x)^{-1} \leftarrow \int u'dx \leftarrow u' = (3+x)^{-2}$

$v = 6x \xrightarrow{\frac{du}{dx}} v' = 6$

Reverse chain rule: divide by the derivative of the bracket

$$\int_{0.2}^{0.4} 6x(3+x)^{-2}\,dx$$

$$= vu - \int uv'\,dx \quad \text{M1}$$

$$= -6x(3+x)^{-1} - \int -(3+x)^{-1} \times 6\,dx$$

$$= -\frac{6x}{3+x} + \int \frac{6}{3+x}\,dx$$



P 7 2 8 0 5 A 0 3 9 4 8

**Question 13 continued**

$$= \left[ -\frac{6x}{3+x} + 6\ln(3+x) \right]_{0.2}^{0.4}$$ M1 A1

$$= \left(6\ln(3.4) - \frac{6(0.4)}{3.4}\right) - \left(6\ln(3.2) - \frac{6(0.2)}{3.2}\right)$$ ddM1

$$= 6\ln\left(\frac{17}{5}\right) - 6\ln\left(\frac{16}{5}\right) - \frac{12}{17} + \frac{3}{8}$$ Log law: $\ln a - \ln b = \ln\frac{a}{b}$

$$= 6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$$ A1

Method 2 SDI – table

| Sign | Differentiate | Integrate |
|---|---|---|
| + | $6x$ | $(3+x)^{-2}$ |
| − | $6$ | $-(3+x)^{-1}$ |
| + | $0$ | $-\ln(3+x)$ |

M1

$$\therefore \int_{0.2}^{0.4} 6x(3+x)^{-2}\,dx$$

$$= -6x(3+x)^{-1} - -6\ln(3+x)$$

$$= \left[6\ln(3+x) - \frac{6x}{3+x}\right]_{0.2}^{0.4}$$

$$= \left[-\frac{6x}{3+x} + 6\ln(3+x)\right]_{0.2}^{0.4}$$ M1 A1

$$= \left(6\ln(3.4) - \frac{6(0.4)}{3.4}\right) - \left(6\ln(3.2) - \frac{6(0.2)}{3.2}\right)$$ ddM1

Log law: $\ln a - \ln b = \ln\frac{a}{b}$ $$= 6\ln\left(\frac{17}{5}\right) - 6\ln\left(\frac{16}{5}\right) - \frac{12}{17} + \frac{3}{8}$$

$$= 6\ln\left(\frac{17}{16}\right) - \frac{45}{136}$$ A1

$$a = 6,\ b = \frac{17}{16},\ c = -\frac{45}{136}$$

DO NOT WRITE IN THIS AREA

**Question 13 continued**

**(Total for Question 13 is 13 marks)**

**14.** **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(a) Show that the equation

$$2\tan\theta(8\cos\theta + 23\sin^2\theta) = 8\sin 2\theta(1 + \tan^2\theta)$$

may be written as

$$\sin 2\theta(A\cos^2\theta + B\cos\theta + C) = 0$$

where $A$, $B$ and $C$ are constants to be found.

**(3)**

(b) Hence, solve for $360° \leqslant x \leqslant 540°$

$$2\tan x(8\cos x + 23\sin^2 x) = 8\sin 2x(1 + \tan^2 x) \qquad x \in \mathbb{R} \qquad x \neq 450°$$

**(4)**

(a) $2\tan\theta(8\cos\theta + 23\sin^2\theta) = 8\sin 2\theta(1-\tan^2\theta)$

Identity: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ B1

Identity: $\cos^2\theta + \sin^2\theta = 1$

double angle formula: $\sin 2\theta = 2\sin\theta\cos\theta$

Identity: $\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$ → $1 + \tan^2\theta = \sec^2\theta$

substitute identities $2\frac{\sin\theta}{\cos\theta}(8\cos\theta + 23(1-\cos^2\theta)) = 8 \times 2\sin\theta\cos\theta\sec^2\theta$

$\sec\theta = \frac{1}{\cos\theta}$

$2\frac{\sin\theta}{\cos\theta}(8\cos\theta + 23(1-\cos^2\theta)) = 8 \times 2\sin\theta\cos\theta \times \frac{1}{\cos^2\theta}$

$2\cos^2\theta \times \frac{\sin\theta}{\cos\theta}(8\cos\theta + 23(1-\cos^2\theta)) = 8 \times 2\sin\theta\cos\theta$ ($2\sin\theta\cos\theta = \sin 2\theta$)

$2\cos\theta\sin\theta\,(8\cos\theta + 23(1-\cos^2\theta)) = 8\sin 2\theta$ ($2\cos\theta\sin\theta = \sin 2\theta$)

Subtract $8\sin 2\theta$ from both sides $\sin 2\theta(8\cos\theta + 23(1-\cos^2\theta)) - 8\sin 2\theta = 0$

factor out $\sin 2\theta$ $\sin 2\theta(8\cos\theta + 23 - 23\cos^2\theta - 8) = 0$

$\sin 2\theta(-23\cos^2\theta + 8\cos\theta + 15) = 0$

divide both sides by $-1$ $\sin 2\theta(23\cos^2\theta - 8\cos\theta - 15) = 0$ hence shown M1A1



P 7 2 8 0 5 A 0 4 2 4 8

**Question 14 continued**

(b) Use the result from (a): $\sin 2x(23\cos^2 x - 8\cos x - 15) = 0$

$\sin 2x = 0 \rightarrow \sin^{-1}(0) = 2$ $\qquad 23\cos^2 x - 8\cos x - 15 = 0$

$2x = 0,\ x = 0, x = -180$ $\qquad (23\cos x + 15)(\cos x - 1) = 0$

add 360° to get in-range values: $0 + 360 = 360°$; $-180 + 360 + 360 = 540$

M1 $\cos x = -\frac{15}{23}$ $\quad x = \cos^{-1}\left(-\frac{15}{23}\right)$ $\quad \cos x = 1$

$x = 130.7 + 360$ $\qquad \cos^{-1}(1) = x = 0 + 360 = 360° = x_1$

$\therefore x_1 = 360,\ x_2 = 540°$ B1 $\qquad x_3 = 490.7$ dM1

Solutions: $x = 360°, 491°, 540°$ to 3sf A1



Turn over

**Question 14 continued**

**Question 14 continued**

**(Total for Question 14 is 7 marks)**

**15.** A student attempts to answer the following question:

> Given that $x$ is an obtuse angle, use algebra to prove by contradiction that
>
> $$\sin x - \cos x \geqslant 1$$

The student starts the proof with:

> Assume that $\sin x - \cos x < 1$ when $x$ is an obtuse angle
>
> $$\Rightarrow (\sin x - \cos x)^2 < 1$$
>
> $$\Rightarrow \ldots$$

The start of the student's proof is reprinted below.

Complete the proof.

**(3)**

Assume that $\sin x - \cos x < 1$ when $x$ is an obtuse angle

$$\Rightarrow (\sin x - \cos x)^2 < 1$$

$\sin^2 x - 2\sin x\cos x + \cos^2 x < 1$ M1

$\sin^2 x + \cos^2 x - 2\sin x\cos x < 1$

use Identity $\sin^2 x + \cos^2 x = 1$ — use double angle formula $2\sin x\cos x = \sin 2x$

$1 - \sin 2x < 1$

$-\sin 2x < 0$ A1

If $x$ is obtuse, it's in the 2nd Quadrant and $2x$ is in the third or fourth quadrant, so $\sin 2x$ must be negative and $\therefore -\sin 2x$ is positive. This is a contradiction with $-\sin 2x < 0$.

$\therefore \sin x - \cos x \geq 1$ By Contradiction A1

DO NOT WRITE IN THIS AREA

**Question 15 continued**

**Question 15 continued**

**(Total for Question 15 is 3 marks)**

**TOTAL FOR PAPER IS 100 MARKS**